PRE REGIONAL MATHEMATICAL OLYMPIAD (PRMO) - 2019

Date: 25/08/2019

Max. Marks: 102 **SOLUTIONS** **Time allowed: 3 hours**

1. Consider the sequence of numbers $\left[n+\sqrt{2n}+\frac{1}{2}\right]$ for $n \geq 1$, where [x] denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < .....$ then find n_{12}.

Ans. (78)

Sol. $\left[n+\sqrt{2n}+\frac{1}{2}\right] = \left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]$

Let $P = \left[\left(\sqrt{n}+0.7\right)^2\right] \rightarrow$ GIF

Given $(n \geq 1)$, put $n = 1 \rightarrow P = 2$

$n = 2 \rightarrow P = 4$

$n = 3 \rightarrow P = 5$

$n = 4 \rightarrow P = 7$

$n = 5 \rightarrow P = 8$

$n = 6 \rightarrow P = 9$

$n = 7 \rightarrow P = 11$

Here we can see that missing number are

1, 3, 6, 10, ...

which is following a certain pattern

Missing Number , $\left(1 \overset{+2}{\frown} 3 \overset{+3}{\frown} 6 \overset{+4}{\frown} 10 \overset{+5}{\frown} 15 \overset{+6}{\frown} 21 \overset{+7}{\frown} 28,\ 36,\ 45,\ 55,\ 66,\ 78\right)$

Hence $n_{12} = 78$

Note : It should be given that n must be postive integer otherwise the questions is bonus.

2. If $x = \sqrt{2}+\sqrt{3}+\sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

Ans. (93)

Sol. $x = \sqrt{2}+\sqrt{3}+\sqrt{6}$

$x-\sqrt{6} = \sqrt{2}+\sqrt{3}$

$\left(x-\sqrt{6}\right)^2 = \left(\sqrt{2}+\sqrt{3}\right)^2$

$x^2 + 6 - 2\sqrt{6}x = 5 + 2\sqrt{6}$

$x^2 + 6 - 5 = 2\sqrt{6}x + 2\sqrt{6}$

$x^2 + 1 = 2\sqrt{6}(x+1)$

$(x^2 + 1)^2 = 24(x^2 + 2x + 1)$

$x^4 + 1 + 2x^2 = 24x^2 + 48x + 24$

$\Rightarrow \quad x^4 - 22x^2 - 48x - 23 = 0$

On comparing with equation, $x^4 + ax^3 + bx^2 + cx + d = 0$ we get,

$a = 0, b = -22, c = -48, d = -23$

$\therefore \quad |a + b + c + d| = |0 - 22 - 48 - 23| = 93.$

3. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Ans. (25)

Sol. Note that every odd number less than 101 can be written as

$$(k + 1)^2 - (k)^2 = 2k + 1$$

Thus now note that, if any even number can be written as difference of two perfect square, then that number must be a multiple of 4, because

$2k = a^2 - b^2 = (a + b)(a - b) \equiv 0 \pmod 4$ if a and b are of same parity.

Also note that, every multiple of 4 can be written as

$$(k + 1)^2 - (k - 1)^2 = 4k$$

Hence total number of numbers less than 101 of the form $a^2 - b^2$ are $50 + 25 = 75$.

Hence the answer is 25.

4. Let $a_1 = 24$ and form the sequence a_n, $n \geq 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

$$24, 2534, 253534, 25353534, \ldots.$$

What is the least value of n for which a_n is divisible by 99?

Ans. (88)

Sol. Note that, every term in the sequence is of the form

$$a_n = 2(53)(53) \ldots (53)4$$

where the number of 53 is the number is $n - 1$. Also if $99|a_n \Leftrightarrow 9|a_n$ and $11|a_n$. Thus, by divisibility rule of 9 we get

$$0 \equiv a_n \equiv 2 + 8(n - 1) + 4 \pmod 9 \Leftrightarrow n \equiv 7 \pmod 9$$

Now by divisibility of 11 we get,

$$0 \equiv a_n \equiv (2 + 3(n - 1)) - (5(n - 1) + 4) \pmod{11} \Leftrightarrow n \equiv 0 \pmod{11}$$

But the minimum solution to the congruences $a_n \equiv 7 \pmod 9$

and $a_n \equiv 0 \pmod{11}$ is 88.

Thus, $n = 88$.

5. Let N be the smallest positive integer such that N + 2N + 3N + ... + 9N is a number all whose digits are equal. What is the sum of the digits of N?

Ans. (37)

Sol. N + 2N + 3N + -------- 9N

= N (1 + 2 + 3 + ...+ 9)

$$= N \times \frac{9\times 10}{2} = 45 \times N$$

We have to multiply with '45' to a number such that, the resulting number should have all digits same.

Such N = 12345679

As 45 × 12345679 = 555555555

$\therefore$ Sum of digits of N = 37

6. Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of ΔABC at B is perpendicular to AC. Find $\angle ABC$ measured in degrees.

Ans. (30)

Sol. Let tangent at B intersect AC at P. Now,

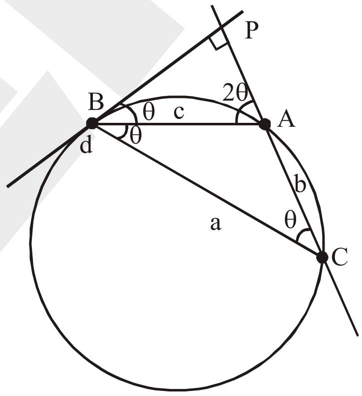
note that $\angle ABC = \angle ACB = \theta \Rightarrow \angle BAP = 2\theta$

$\Rightarrow \angle PBA = \theta$,

In ΔPBA

$3\theta = 90°$

$\theta = 30°$



7. Let s(n) denote the sum of the digits of a positive integer n in base 10. If s(m) = 20 and s(33m) = 120, what is the value of s(3m)?

Ans. (60)

Sol. We will take sum of digit base 10 to (mod 9)

Also s(ab) = s(a). s(b) (mod9)

Now s(m) = 20

s(33 m) = 120 = s(11) × s(3m)

120 = 2 × s(3m) [$\because$ s(11) = 2(mod 9)]

60 = s(3m)

Hence, s(3m) = 60

8. Let $F_k(a, b) = (a + b)^k - a^k - b^k$ and let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For how many ordered pairs (a, b) with $a, b \in S$ and $a \le b$ is $\frac{F_5(a,b)}{F_3(a,b)}$ an integer?

Ans. **(22)**

Sol. $\frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$

$$\Rightarrow \frac{a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^4 - a^5 - b^5}{a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3}$$

$$\Rightarrow \frac{5ab[a^3 + 2a^2b + 2ab^2 + b^3]}{3ab[a+b]}$$

$$\Rightarrow \frac{5}{3}\frac{[a^3 + 3a^2b + 3ab^2 + b^3 - a^2b - ab^2]}{(a+b)}$$

$$\Rightarrow \frac{5}{3}\frac{[(a+b)^3 - ab(a+b)]}{(a+b)}$$

$$\Rightarrow \frac{5}{3}\frac{[(a+b)\{(a+b)^2 - ab\}]}{(a+b)}$$

$$\Rightarrow \frac{5}{3}[a^2 + 2ab + b^2 - ab]$$

$$\Rightarrow \frac{5}{3}[a^2 + ab + b^2]$$

Now $3 | a^2 + ab + b^2$

Both a and b cannot simultaneously be even.

Three cases are possible

Case 1 : $a \equiv 0 \pmod 3$

$b \equiv 0 \pmod 3$

$\therefore$ (a, b) = (3, 3); (6, 6); (9, 9) $\Rightarrow$ 3 pairs

Case 2 : $a \equiv 1 \pmod 3$

$b \equiv 1 \pmod 3$

$\therefore$ (a, b) = (1, 1); (1, 4); (1, 7); (1, 10);
(4, 4); (4, 7); (4, 10);
(7, 7); (7, 10);
(10, 10) $\Rightarrow$ 10 pairs

Case 3 : $a \equiv 2 \pmod 3$

$b \equiv 2 \pmod 3$

$\therefore$ (a, b) = (2, 2); (2, 5); (2, 8) $\Rightarrow$ 6 pairs
(5, 5); (5, 8);
(8, 8)

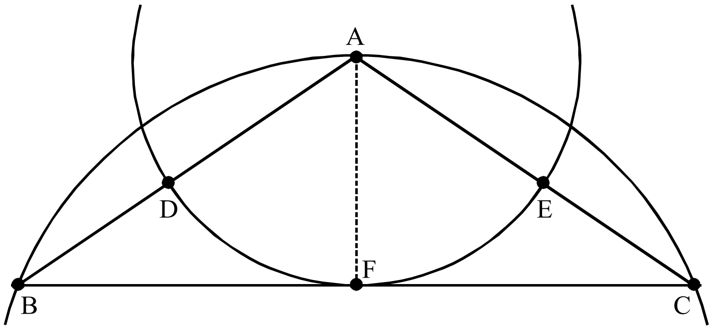
and (3, 6); (3, 9); (6, 9) $\Rightarrow$ 3 pairs

$\therefore$ total = 3 + 10 + 6 + 3 = 22 pairs

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is $\alpha°$ and the smaller one $\beta°$ then what is the value of $\alpha - \beta$?

Ans. (90)

Sol.



The center of any circle wtih D and E on it must pass through the (potentially extended) bisector of $\angle A$. For this center to be on the circumcircle of ΔABC, the only possibility is for the center to be A itself.

AF = AD since they are both radii of the same circle.

AD = BD since D is the midpoint of $\overline{AB}$.

$AF \perp BC$, since ΔABC is isosceles.

Therefore, since AB = 2AF, $\angle B = 30°$. That makes $\angle A = 120°$, so the difference between them is 90°.

10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

Ans. (42)

Sol. Let x = ab

where a and b are the unit digits

Time after 5'O clock = $5 \times 60 + 10a + b$

$= (300 + 10a + b)$ minutes

Time after 6'O clock = $6 \times 60 + 10b + a$

$= (360 + 10b + a)$ minutes

$\therefore \quad 360 + 10b + a - 300 - 10a - b = 10a + b$

$60 + 9b - 9a = 10a + b$

$60 + 8b = 19a$

$\therefore \quad b = \dfrac{19a - 60}{8}$

Now a & b are integers from (1 to 9) by putting different values of a, we get a = 4, b = 2

$\therefore \quad x = 42$ minutes.

11. Find the largest value of a^b such that the positive integers a, b > 1 satisfy $a^b b^a + a^b + b^a = 5329$.

Ans. (81)

Sol. $a, b > 1$

$a^b b^a + a^b + b^a = 5329$

$a^b b^a + a^b + b^a + 1 = 5330$

$(a^b + 1)(b^a + 1) = 2 \times 5 \times 13 \times 41$

$= 65 \times 82$

$a^b + 1 = 82$

$a^b = 81 = 3^4$

$a = 3, b = 4$

Now $b^a = 4^3 = 64$ as $b^a + 1 = 65$

$\therefore$ Largest values of $a^b = 81$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$$\{10a + b : 1 \le a \le 5, 1 \le b \le 5\}$$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Ans. (47)

Sol. $10a + b$; $1 \le a \le 5, 1 \le b \le 5$

Let us divide numbers into different sets, such as

Set 1 : {11, 12, 13, 14, 15}

Set 2 : {21, 22, 23, 24, 25}

Set 3 : {31, 32, 33, 34, 35}

Set 4 : {41, 42, 43, 44, 45}

Set 5 : {51, 52, 53, 54, 55}

Now to make a number having no two unit's digit and no two ten's digit same, we can select any 1 number from each of set 1, set 2, set 3, set 4, set 5

$\therefore$ Number of ways : $5_{C_1} \times 4_{C_1} \times 3_{C_1} \times 2_{C_1} \times 1_{C_1}$

$= 5! = 120$

$\therefore$ $120 \div 73 \Rightarrow$ Remainder = 47

13. Consider the sequence

$$1, 7, 8, 49, 50, 56, 57, 343, ...$$

which consists of sums of distinct powers of 7, that is $7^0, 7^1, 7^0 + 7^1, 7^2$, ..., in increasing order. At what position will 16856 occur in this sequence?

Ans. (36)

Sol. Serial number Binary $(\text{Base})_7$ Series

Serial number	Binary		$(\text{Base})_7$	Series
1	$(1)_2$	$\longrightarrow$	$(1)_7$	$7^0 = 1$
2	$(10)_2$	$\longrightarrow$	$(10)_7$	$7^1 = 1$
3	$(11)_2$	$\longrightarrow$	$(11)_7$	$7^1 + 7^0 = 8$
4	$(100)_2$	$\longrightarrow$	$(100)_7$	$7^2 = 49$
5	$(101)_2$	$\longrightarrow$	$(101)_7$	$7^2 + 7^0 = 50$
6	$(110)_2$	$\longrightarrow$	$(110)_7$	$7^2 + 7^1 = 56$

7	16856	
7	2408	0
7	344	0
7	49	1
7	7	0
	1	0

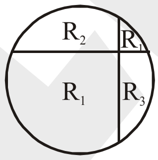
$(16856) = (100\ 100)_7\ (100\ 100)_2 = 2^5 + 2^2 = 36^{th}$

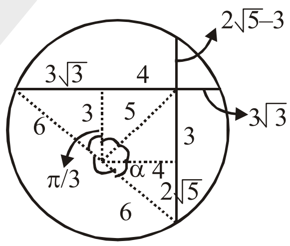
$\therefore$ 16856 is 36^{th} term.

14. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide R into four regions R_i, $i = 1, 2, 3, 4$. If $|R_i|$ denotes the area of the region R_i and if $[R_1] > [R_2] > [R_3] > [R_4]$, determine $[R_1] - [R_2] - [R_3] + [R_4]$. (Here $[\Omega]$ denotes the area of the region Ω in the plane.)

Ans. (48)

Sol. We have $\Sigma[R_1] = 36\pi$





$[R_1] - [R_2] - [R_3] + [R_4]$

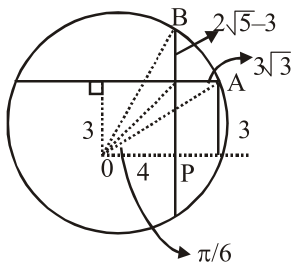
$= 2([R_1] - [R_2])-\Sigma[R_1]$

$= 2([R_1] + [R_4]) - 36$

Now $\cos\alpha = \dfrac{2}{3}$

Also, $\theta = \dfrac{7\pi}{6} - \alpha$

$[R_1] = \dfrac{1}{2} \times 36 \times \left(\dfrac{7\pi}{6} - \alpha\right) + \dfrac{1}{2}(4 + 3\sqrt{3}) \times 3 + \dfrac{1}{2} \times (3 + 2\sqrt{5}) \times 4$



Now $\angle BOP = \alpha$ as $\cos\alpha = \dfrac{2}{3}$

$\Rightarrow \quad \angle BOA = \alpha - \dfrac{\pi}{6}$

so, $[R_4] = \dfrac{1}{2} \times 36\left(\alpha - \dfrac{\pi}{6}\right) - \dfrac{1}{2} \times (2\sqrt{5} - 3) \times 4 - \dfrac{1}{2}(3\sqrt{3} - 4) \times 3$

Thus $[R_1] + [R_4] = 18\pi + 12 + 12 = 24 + 18\pi$

Som required answer

$= 48$

15. In base –2 notation, digits are 0 and 1 only and the places go up in powers of –2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base –2 how many non zero digits does it contain?

Ans. **(06)**

Sol. $2019 = 2048 - 32 + 4 - 2 + 1$

$= (-2)^{12} + (-2)^{11} + (-2)^5 + (-2)^1 + (-2)^0$

$= 4096 - 2048 - 32 + 4 - 2 + 1$

$= 1100000100111$ (in base –2)

Number of non-zero digits = 6

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

Ans. **(69)**

Sol. n = natural number.

p = prime number

$p < 20$

$\therefore \quad p^2 < 400$

also $\sqrt{n} < p$

$\therefore \quad n < p^2 < 400$

so $\quad n \in \{1, 2, ... 399\}$

If p = 2, then $n < 2^2 \quad \Rightarrow \quad n < 4$

$\therefore \quad n = 2$ only case $\quad \Rightarrow \quad$ 1 solution

If p = 3, then $n < 3^2 \quad \Rightarrow \quad n < 9$

$\therefore \quad n = 3, n = 6 \quad \Rightarrow \quad$ 2 solution

If p = 5, then $n < 5^2 \quad \Rightarrow \quad n < 25$

$\therefore \quad n = 5, 10, 15, 20 \quad \Rightarrow \quad$ 4 solution

If p = 7, then $n < 7^2 \quad \Rightarrow \quad n < 49$

$\therefore \quad n = 7, 14, 21, 28, 35, 42 \quad \Rightarrow \quad$ 6 solution

If p = 11, then $n < 11^2 \quad \Rightarrow \quad n < 121$

$\therefore \quad n = 11, 22, 33, 44, 55, 66, 77, 88, 99, 110 \quad \Rightarrow \quad$ 10 solution.

If p = 13, then $n < 13^2 \quad \Rightarrow \quad n < 169$

$\therefore \quad n = 13, 26, 39, ..., 156 \quad \Rightarrow \quad$ 12 solution.

If p = 17, then $n < 17^2 \quad \Rightarrow \quad n < 289$

$\therefore \quad n = 17, 34, ..., 272 \quad \Rightarrow \quad$ 16 solution.

If p = 19, then $n < 19^2 \quad \Rightarrow \quad n < 361$

$\therefore \quad n = 19, 38, ..., 342 \quad \Rightarrow \quad$ 18 solution.

Total $1 + 2 + 4 + 6 + 10 + 12 + 16 + 18 \Rightarrow 69$

17. Let a, b, c be distinct positive integers such that $b + c - a$, $c + a - b$ and $a + b - c$ are all perfect squares. What is the largest possible value of $a + b + c$ smaller than 100?

Ans. **(91)**

Sol. Let $b + c - a = x^2$ ... (i)

$c + a - b = y^2$ ... (ii)

$a + b - c = z^2$ ... (iii)

Now since a, b, c are distinct positive integers,

$\therefore$ x, y, z will also be positive integers,

add (i), (ii) and (iii)

$a + b + c = x^2 + y^2 + z^2$

Now, we need to find largest value of $a + b + c$ or $x^2 + y^2 + z^2$ less than 100.

Now, to get a, b, c all integers x, y, z all must be of same parity, i.e. either all three are even or all three are odd.

Let us maximize $x^2 + y^2 + z^2$, for both cases.

If x, y, z are all even.

$\Rightarrow$ $b + c - a = 8^2 = 64$

$c + a - b = 4^2 = 16$

$a + b - c = 2^2 = 4$

Which on solving, give $a = 10$, $b = 34$, $c = 40$ and $a + b + c = 84$

If x, y, z are all odd

$\Rightarrow$ $b + c - a = 9^2 = 81$

$c + a - b = 3^2 = 9$

$a + b - c = 1^2 = 1$

Which on solving, give $a = 5$, $b = 41$, $c = 45$ and $a + b + c = 91$

$\therefore$ Maximum value of $a + b + c < 100 = 91$

18. What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?

Ans. **(43)**

Sol. $p^3 + 4p^2 + 4p$

$\Rightarrow$ $p(p^2 + 4p + 4)$

$\Rightarrow$ $p(p + 2)^2$

This number has 30 divisors so it can be in the form of

$a.b^{14}$

$a^2 . b^9$

$a. b^2. c^4$

a^{29}

Out of these cases, we will check cases in which p can be minimum which seems to be possible with $a. b^2 . c^4$ case

so by simply checking several cases,

we can put p = 43

$\because \quad \Rightarrow \quad 43\ (45)^2$

$\Rightarrow \quad 43 \times 15^2 \times 3^2$

$\Rightarrow \quad 43 \times 5^2 \times 3^2 + 3^2$

$\Rightarrow \quad 43 \times 5^2 \times 3^4$

Whose no. of divisors are

$(1 + 1) \cdot (2 + 1) \cdot (4 + 1)$

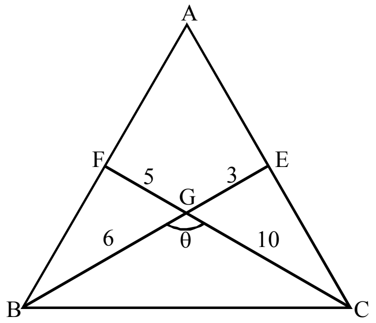
$2 \times 3 \times 5 = 30$

$\therefore \quad p = 43$

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

Ans. (90)

Sol.



Area of $\Delta BGC = \frac{1}{2} \times 6 \times 10 \times \sin\theta$

To maximize the area of ΔBGC, $\sin\theta = 1$

$\therefore$ maximum area of $\Delta BGC = 30$

Maximum area of $\Delta ABC = 3\ \Delta BGC$

$= 3 \times 30 = 90$ sq. units

20. How many 4-digit numbers $\overline{abcd}$ are there such that $a < b < c < d$ and $b - a < c - b < d - c$?

Ans. (07)

Sol. $\overline{abcd}$

$\therefore \quad a < b < c < d$

$\therefore \quad a \geq 1$

$b \geq 2$

$c \geq 3$

$d \geq 4$

Also $\quad b - a < c - b$

i.e. $2b < a + c$

and $\quad c - b < d - c$

$2c < b + d$

We can made a table applying all these conditions

a	b	c	d
1	2	4	7
1	2	4	8
1	2	4	9
1	2	5	9
2	3	5	8
2	3	5	9
3	4	6	9

So total 7 numbers are possible.

21. Consider the set E of all positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

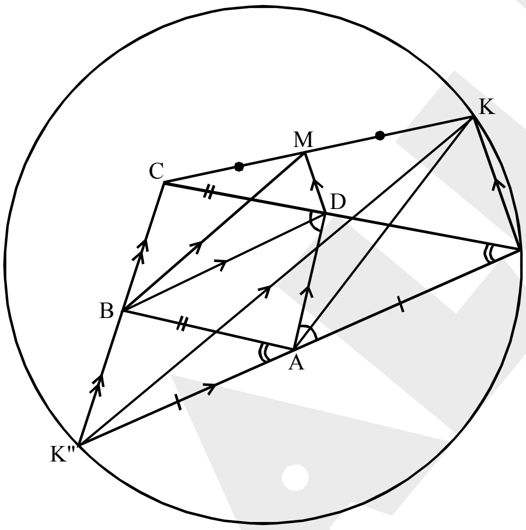
Ans. **(Bonus)**

Sol. Question is incorrect.

22. In parallelogram ABCD, AC = 10 and BD = 28. The points K and L in the plane of ABCD move in such a way that AK = BD and BL = AC. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Ans. **(02)**

Sol.



Produce CD to K' such that CD = DK'

then BDK'A is a parallelogram

$\because$ AB = CD = DK'

AB$\|$DK'

$\therefore$ AK' = BD

Draw a circle with centre A and radius BD which cuts CD produced at K' and CB produced at K"

then K" AK' are collinear as $\angle CDA + \angle BAD = 180°$

$\angle CDA = \angle DAK' + \angle DK'A = \angle DAK' + \angle BAK''$ [$\because$ BA$\|$DK']

$\therefore$ $\angle DAK' + \angle BAK'' + \angle BAD = 180°$

Thus K'AK" is a diameter.

Let K is any point on this circle

Since M is a mid point of CK

D is a mid point of CK'

then MD||KK'

In ΔCK'K",

D is a mid point of CK'

DB||K'A i.e, DB||K'K"

$\therefore$ B is a mid point of CK"

In ΔCK"K

B, M are the mid points of CK" and CK respectively.

$\therefore$ In ΔBMD and ΔK"KK'

BM||K"K

MD||KK'

BD||K"K'

$\therefore$ $\angle$BMD = $\angle$K"KK' = 90°

$\because$ K"K' is a diameter similarly for other Δ

$\angle$ANC = 90°

So $\frac{\angle BMD}{2} = 45°$, $\frac{\angle ANC}{2} = 45°$

$$\cot^2\left(\frac{\angle BMD}{2}\right) + \tan^2\left(\frac{\angle ANC}{2}\right) = 1 + 1 = 2$$

23. Let t be the area of a regular pentagon with each side equal to 1. Let P(x) = 0 be the polynomial equation with least degree, having integer coefficients, satisfied by x = t and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{M}$? ($\sin 18° = (\sqrt{5}-1)/2$).

Ans. **(Bonus) Note :** sin18° value in the question given wrong. Originally $\sin 18° = (\sqrt{5}-1)/4$)

Sol. Area of regular polygon = $\frac{a^2 n}{4\tan\left(\frac{180}{n}\right)}$

n = number of sides

a = length of side

$\therefore$ for regular pentagon of side length 1,

$$\text{area} = t = \frac{5}{4\tan 36°} = \frac{5}{4(0.73)} \cong 1.71$$

Now, P(1.71) = 0 to be found with least degree and integer coefficient such that gcd of all coefficeints is 1.

Let $x = 1.71$

$100x = 171$

$\therefore$ $P(x) = 100x - 171 = 0$ is the required polynomial

Which satisfied all the conditions.

$\therefore$ $m = |100| + |-171| = 271$

$\therefore$ $\sqrt{m} = 16.46$

$\therefore$ nearest integer = 16

But this question can have multiple solutions as student can take tan36° as 0.72, 0.726 or even 0.7, every time we will get different answers. So this question should be Bonus.

24. For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$. Define $f(n) = \max\{m \in N \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + ... + f(10)$.

Ans. **(75)**

Sol. $a_1 = 9744$

$a_2 = 99744$

$a_3 = 999744$

and so on...

$\because$ 9744 is divisible by 16

$\therefore$ Each a_n is divisible by atleast 2^4.

Now, $a_1 = 10^4 - 256 \equiv 0 \pmod{2^4}$

$a_2 = 10^5 - 256 \equiv 0 \pmod{2^5}$

$a_3 = 10^6 - 256 \equiv 0 \pmod{2^6}$

$a_4 = 10^7 - 256 \equiv 0 \pmod{2^7}$

$a_5 = 10^8 - 256 = 256\,(390625 - 1)$

$= 256 \times 390264$

$= 256 \times 32 \times 12207$

$= 2^{13} \times 12207$

$\equiv 0 \pmod{2^{13}}$

$a_6 = 10^9 - 256 \equiv 0 \pmod{2^8}$

$a^7 = 10^{10} - 256 \equiv 0 \pmod{2^8}$

$a_8 = 10^{11} - 256 \equiv 0 \pmod{2^8}$

$a_9 = 10^{12} - 256 \equiv 0 \pmod{2^8}$

$a_{10} = 10^{13} - 256 \equiv 0 \pmod{2^8}$

$\therefore$ $f(1) + f(2) + ... + f(10)$

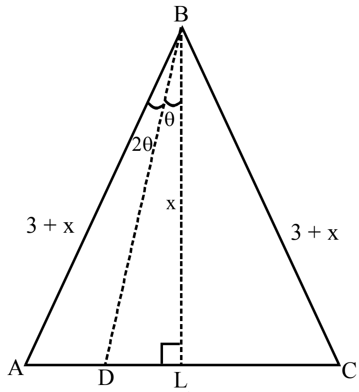
$= 4 + 5 + 6 + 7 + 13 + 8 + 8 + 8 + 8 + 8$

$= 75$

25. Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and AB – BD = 3, find AC.

Ans. **(26)**

Sol. Let $B = 6\theta$



Let $BD = x \in Z$

$\Rightarrow \quad AB = x + 3$

Given $AC \in Z$

$A = (3 + x)\cos 3\theta = x\cos\theta$

$\Rightarrow \quad (3 + x)(4\cos^2\theta - 3) = x$

$\Rightarrow \quad \sin^2\theta = \dfrac{3}{4(x+3)}$

Now $AL = (3 + x)\sin\theta$

$\Rightarrow \quad AC = 2(3 + x)(3\sin\theta - 4\sin^3\theta)$

$= 2(3 + x)\sin\theta\left(3 - 4\dfrac{3}{4(x+3)}\right)$

$= 6(2 + x)\dfrac{\sqrt{3}}{2\sqrt{x+3}} = 3(x + 2)\dfrac{\sqrt{3}}{\sqrt{x+3}}$

$\Rightarrow \quad x = 3y \qquad \Rightarrow \quad AC = \dfrac{3(3y+2)}{\sqrt{y+1}}$

$\Rightarrow \quad y + 1 = z^2 \qquad \Rightarrow \quad AC = \dfrac{3(3z^2-1)}{z} = 9z - \dfrac{3}{z}$

$\Rightarrow \quad z = 1$ or 3

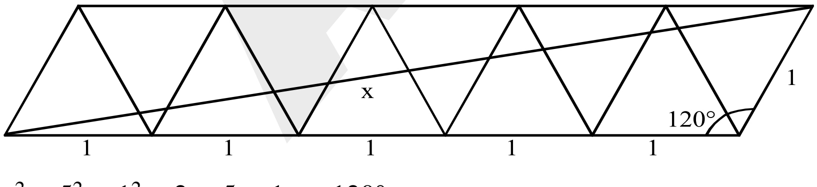
But $z \neq 1$ as $x = 0$ not possible

$\Rightarrow \quad z = 3 \qquad \Rightarrow \quad AC = 26$

26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N?

Ans. **(31)**

Sol.



$x^2 = 5^2 + 1^2 - 2 \times 5 \times 1 \cos 120°$

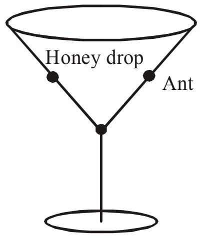
$= 25 + 1 + 5$

$x = \sqrt{31}$

$N = 31$

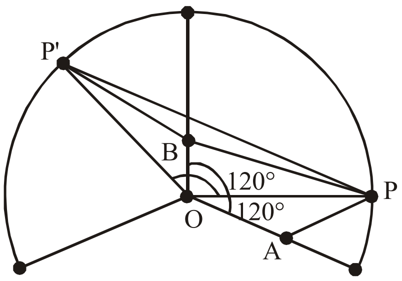
Folding the triangle continously each time of reflection creates the above diagram. 9 points of reflection can be verified in the diagram above. Thus root (N) is the length of red line which is root (31). Thus N = 31 is the answer.

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass (See the figure.). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass.)



Ans. **(36)**

Sol.



Rotate ΔOAP by 120° in anticlockwise then A will be at B, P will be at P' $\Rightarrow \Delta OAP \equiv \Delta OBP'$

$\Rightarrow PB + PA = P'B + PB \geq P'P$

Minimum $PB + PA = P'P$ equality when P on the angle bisector of $\angle AOB$

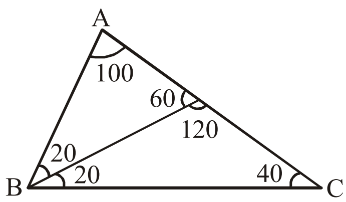
$\Rightarrow P'P = 2(21)\sin 60° = 21\sqrt{3}$

$[\min(PB + PA)] = [21\sqrt{3}\,] = 36$

28. In a triangle ABC , it is known that $\angle A = 100°$ and AB = AC . The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10° \approx 0.174$.

Ans. **(27)**

Sol. Given, BD = 20 units



$\angle A = 100°$

AB = AC

In ΔABD

$$\frac{BD}{\sin A} = \frac{AD}{\sin 2\theta}$$

$$\frac{BD}{\sin 100°} = \frac{AD}{\sin 20°}$$

$$\frac{BD}{\cos 10°} = \frac{AD}{2\sin 10°\cos 20°}$$

$\Rightarrow \quad 20 = \dfrac{AD}{2\sin 10°} \Rightarrow AD = 40.\sin 10° = 6.96$

In ΔBDC

Also, $\dfrac{BD}{\sin 40°} = \dfrac{BC}{\sin 120°} = \dfrac{CD}{\sin 20°}$

$$\frac{20}{2\sin 20°.\cos 20°} = \frac{CD}{\sin 20°} \Rightarrow CD = \frac{20}{2\cos 20°} = \frac{20}{2\times 0.9394} \cong 10.65$$

$\therefore$ AD + CD = AC = AB $\cong$ 17.6

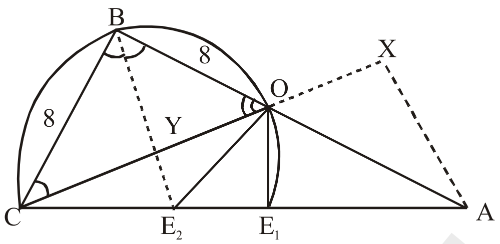
Now since BD is angle bisector

So $\dfrac{BC}{AB} = \dfrac{CD}{AD} \Rightarrow BC = \dfrac{AB\times CD}{AD} = \dfrac{17.6\times 10.65}{6.96} \cong 26.98 \cong 27$

29. Let ABC be an acute angled triangle with AB = 15 and BC = 8. Let D be a point on AB such that BD = BC. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible values of AE, find $[\alpha]$ the integer part of α.

Ans. (68)

Sol.



The pairs E_1, E_2 satisfyies condition or E_1 = intersection of CBO with AC and E_2 = intersection of $\angle$bisector of B and AC

$\because$ that $\angle DE_2B = \angle CE_2B$ and for E_1 $\angle BE_1C = \angle BDC = \angle BCD = \angle BE_1D$)

$\Rightarrow$ $\overline{AE_1}, \overline{AC} = \overline{AD}\cdot\overline{AB} = 7 \times 15$

$$\frac{\overline{AE_2}}{\overline{AC}} = \frac{XY}{XC}$$

(For y is midpoint of OC and X is foot of altitude from A to CD)

Also $\dfrac{XD}{DY} = \dfrac{7}{8}$ and DY = YC

$$\Rightarrow \frac{XD+DY}{XC} = \frac{15}{7+8+8} = \frac{15}{23}$$

$$\Rightarrow \frac{XY}{XC} = \frac{15}{23} \Rightarrow \frac{AE_2}{AC} = \frac{15}{23}$$

$$\Rightarrow AE_1 . A\overline{E}_2 = \frac{15}{23}\cdot 7\cdot 15 = \frac{225\times 7}{23}$$

Ans. $\left[\dfrac{225\times 7}{23}\right] = 68$

30. For any real number x, let [x] denote the integer part of x; {x} be the fractional part of x. ({x} = x − [x]) . Let A denote the set of all real numbers x satisfying

$$\{x\} = \frac{x+[x]+[x+(1/2)]}{20}$$

If S is the sum of all numbers in A, find [S].

Ans. **(21)**

Sol. $\{x\} = \dfrac{x+[x]+\left[x+\frac{1}{2}\right]}{20} \Rightarrow 20\,f = 2I + f + \left[x+\frac{1}{2}\right]$

$$\Rightarrow 19f = 2I + \left[x+\frac{1}{2}\right]$$

Let x = I + F = [x] + {x}

case I: $0 \le f < \frac{1}{2}$ i.e. $\left[x+\frac{1}{2}\right] = I$

so, $19f = 3I \in \left[0,\frac{19}{2}\right)$

$\Rightarrow \quad I \in \left[0,\frac{19}{6}\right)$

Hence, $x = I + f = I + \frac{3I}{19} = \frac{22I}{19}$;

$I = 0, 1, 2, 3$

Case II: $f \in \left[\frac{1}{2},1\right)$ i.e. $\left[x+\frac{1}{2}\right] = I + 1$

so, $19f = 3I + 1 \in \left[\frac{19}{2},19\right)$

$\Rightarrow I \in \left[\frac{17}{6},6\right)$

$\Rightarrow \quad I = 3, 4, 5$

Hence, $x = I + f = I + \frac{3I+1}{19}$

$= \frac{22I}{19} + \frac{1}{19}$; $I = 3, 4, 5$

Thus, $S = \frac{22}{19} \times 18 + \frac{3}{19} = 21$

$[S] = 21$